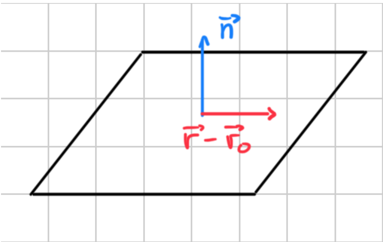
Section 3: Equation of Planes

For equation of planes we need:

- a normal vector (perpendicular vector)
- a point



$\vec{r} = \langle x, y, z \rangle$

$\vec{n} = \langle a, b, c \rangle$

$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Simplified: $ax + by + cz = d$

Example 1: Find the equation of the plane through $(2, 1, 4)$ perpendicular to $\langle -3, 1, 2 \rangle$.

$\langle x_0, y_0, z_0 \rangle = (2, 1, 4)$ and $\langle a, b, c \rangle = \langle -3, 1, 2 \rangle$

$-3(x - 2) + (y - 1) + 2(z - 4) = 0$

$-3x + 6 + y - 1 + 2z - 8 = 0$

$-3x + y + 2z = 3$

Example 2: Find the equation of the plane containing $P(0, 1, 1), Q(3, 2, 1), \text{and } R(1, 0, 0)$.

Use the cross product to find $\vec{n}$

$\vec{n} = \vec{PQ} \times \vec{PR}$

$\vec{PQ} = \langle 3 - 0, 2 - 1, 1 - 1 \rangle = \langle 3, 1, 0 \rangle$ $\vec{PR} = \langle -2, -1, -1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{vmatrix} = \langle -1, 3, -1 \rangle$$

$\langle a, b, c \rangle = \langle -1, 3, -1 \rangle$

$-1(x-0)+3(y-1)-(z-1)=0$

$-x+3y-3-z+1=0$

$-x+3y-z=2$

Example 3: Find the equation of the plane through $(2,-1,4)$ perpendicular to the line

$x=3+4t, \quad y=2+7t, \quad z=-t$

$\vec{v}=\langle 4,7,-1\rangle$

$4(x-2)+7(y+1)-(z-4)=0$

Parallel planes have the same normal vector.

Example 4: Find the equation of the plane that contains the line

$x=2+t, \quad y=3-2t, \quad z=1+5t$

and is parallel to the plane $3x+5y+2z=10$

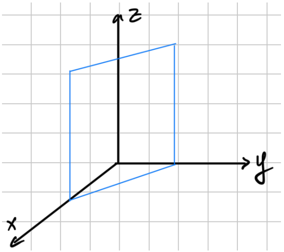
$\vec{n}=\langle 3,5,2\rangle \qquad \langle x_0,y_0,z_0\rangle=(2,3,1)$

$3(x-2)+5(y-3)+2(z-1)=0$

Graph planes

$2x+3y+z=6$

$z=6-2x-3y$



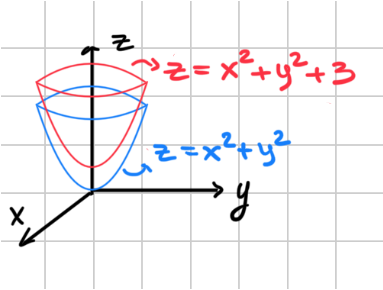
Intercepts:

$x=0, y=0, z=6$

$y=0, z=0, x=3$

$x=0, z=0, y=2$

Paraboloids



$z = x^2 + y^2$

$z = x^2 + y^2 + 3$

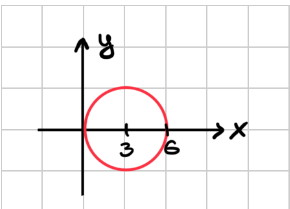
- If coefficients are the same, it is a **sphere**.
- If coefficients are not the same, it is an **ellipsoid.**

$ax^2 + by^2 + cz^2 = d$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Plane: $ax + by + cz = d$

Example 5: Turn the equation $(x-3)^2 + y^2 = 9$ into a polar equation then identify and sketch.



$(x-3)^2 + y^2 = 9$ → circle with $r = 9$

$x^2 - 6x + 9 + y^2 = 9$

$x^2 - 6x + y^2 = 0$

$x^2 + y^2 = 6x$

$r^2 = 6r\cos\theta$

$r = 6\cos\theta$ → far distance

Example 6: Determine if L is parallel or perpendicular to P.

L: $x = 1 + 3t$

$y = 4 - t$

$z = 2 - 3t$

P: $2x + 3y + z = 10$

$\vec{v_1} = \langle 3, -1, -3 \rangle$

$\vec{n} = \langle 2, 3, 1 \rangle$

$\vec{n} \cdot \vec{v_1} = 6 - 3 - 3 = 0$

→ because the normal vector is perpendicular the line is parallel to the plane.

Example 7: Determine if lines are intersecting, skew or neither.

$L_1 : x = 2 - t_1$ $\quad$ L_2: $x = 4 - t_2$

$y = 4 + 2t_1$ $\quad$ $y = 3 + t_2$

$z = -5 + t_1$ $\quad$ $z = -13 + 3t_2$

$\vec{v_1} = \langle -1, 2, 1 \rangle$

$\vec{v_2} = \langle -1, 1, 3 \rangle$

$\vec{v_1} \cdot \vec{v_2} = 1 + 2 + 3 = 6$

→ not 0 so no info

Check if it is Intersecting:

$2 + t_1 = 4 - t_2$ $\quad$ $2 - t_1 = 4 - t_2$

$4 + 2t_1 = 3 + t_2$ $\quad$ $4 + 2t_1 = 3 + t_2$

$-5 + t_1 = -13 + 3t_2$ $\quad$ → Add: $6 + t_1 = 7$

$t_1 = 1$

$2 - 1 = 4 - t_2$

$t_2 = 3$

$-5 + 1 = -13 + 3 \cdot 3$

Lines are **intersecting**.

Where? $x = 1, \quad y = 6, \quad z = -4$

Example 8: Find values of a and c so that $(a, 1, c)$ lies on the line through $P(0, 2, 3)$ and $Q(2, 7, 5)$.

$\vec{PQ} = \langle 2, 5, 2 \rangle$

$x = 2t$ $\quad$ $a = 2t$

$y = 2 + 5t$ $\quad$ → $\quad$ $1 = 2 + 5t$

$z = 3 + 2t$ $\quad$ $c = 3 + 2t$

$5t = -1$

$t = -\frac{1}{5}$

$a = 2t = -\frac{2}{5}$

$c = 3 + 2t = 3 - \frac{2}{5} = \frac{13}{5}$

Example 9: Find the equation of the plane through $(1, -2, 3)$ parallel to the xz-plane.

$\langle 0, 1, 0 \rangle$ is the perpendicular vector to the xz-plane.

$0(x-1) + 1(y+2) + 0(z-3) = 0$
$y + 2 = 0$