Classic Middle-Grades Problems for the Classroom

This lesson presents two classic problems (Mangoes Problem and Sailors and Coconuts) that can be represented and solved in several different ways. Middle-grades students work in groups on the problems to promote communication of mathematical ideas, and a variety of classroom solution attempts are described. This lesson plan was adapted from an article, written by Jerry Stonewater, which appeared in the November‑December 1994 issue of Mathematics Teaching in the Middle School.

Learning Objectives:

- apply and adapt a variety of appropriate strategies to solve problems
- monitor and reflect on the process of mathematical problem solving
- communicate their mathematical thinking coherently and clearly to others

Materials

- The Mangoes Problem Activity Sheet
- The Sailors and Coconuts Problem Activity Sheet

Instructional Plan

The Mangoes Problem

Read the mangoes problem to the students.

One night the King couldn't sleep, so he went down into the Royal kitchen, where he found a bowl full of mangoes. Being hungry, he took 1/6 of the mangoes.

Later that same night, the Queen was hungry and couldn't sleep. She, too, found the mangoes and took 1/5 of what the King had left.

Still later, the first Prince awoke, went to the kitchen, and ate 1/4 of the remaining mangoes.

Even later, his brother, the second Prince, ate 1/3 of what was then left.

Finally, the third Prince ate 1/2 of what was left, leaving only three mangoes for the servants.

How many mangoes were originally in the bowl?

Before students actually solve the problem, ask them to discuss, in groups, possible strategies for solving the problem. Possible strategies include:

- guess and check
- draw a picture
- work backward
- write an equation (use a variable)

Distribute the The Mangoes Problem activity sheet so students may see the text of the entire problem and have a place to show their work.

The Mangoes Problem Activity Sheet

This problem can also be used for a variety of instructional purposes, including assessment, where the focus might be on assessing students' ability to use a variety of strategies, or as a task for a cooperative-problem-solving group, where the goal is to use as many different strategies as possible in solving the problem.

The following sections:

- outline each of the four previously mentioned solution methods for the mangoes problem,
- discuss how middle school students approached the problem, and
- suggest two interesting generalizations of the problem.

Guess and check: The guess-and-check strategy starts with an original guess for how many mangoes were in the bowl prior to the King's entry into the kitchen. Students then use the structure of the problem to see if their initial guess works to solve the problem correctly. If their initial guess fails to work, they make another, it is hoped "better," guess and check to see if it works. They continue this process until they make a correct guess. Some students may make wild and unreasonable guesses, so teachers should point out how to make "reasonable" first guesses and discuss the importance of making a table to collect and organize the data.

Students might realize that an initial guess has to be divisible by 6 so that the King could take one-sixth of the mangoes. For example, a student might guess that 24 mangoes were in the bowl originally. When checking this guess, however, the student will find that it results in 4, not 3, mangoes at the end. Since this outcome is too many mangoes, the student would revise his or her initial guess downward to 18, the next smallest multiple of 6. This number does, in fact, work.

Not all students will necessary note the relevance of the initial guess's being a multiple of 6. An initial guess may be 14, suggesting that students are not aware of the relevance of divisibility by 6. For their guess of 14, students may get off track and do the following computation on a calculator:

14 - 1/2 - 1/3 - 1/4 - 1/5 - 1/6

Draw a picture: The easiest solution method to this problem is surprising in its simplicity. Start by drawing a rectangle to represent all mangoes in the original pile prior to the removal of any of them. Since the King took one-sixth of this pile, divide the rectangle into six equal strips and "remove" one strip. Notice that five strips remain, from which the Queen removed one-fifth, so this one-fifth is also represented by one of the original strips. Continuing, when the first Prince removes one-fourth of what is left, the one-fourth is represented by one of the strips. Similarly, the one-third, one-half, and 3 remaining mangoes are each represented by a strip. In the final analysis, since the 3 mangoes equal one strip and originally six strips were involved, the number of original mangoes must have been 6 × 3 = 18.

The draw-a-picture strategy presents a nice concrete, visual representation of the problem.

The draw-a-picture strategy may lead to some of your most interesting observations. Students may first draw six circles and shaded one to represent the one-sixth the King took. They then would explain that the Queen ate one-fifth of what was left, so they would have shaded one of the remaining five circles. The process is continued until students have shaded the last of the original six circles drawn. Since the last circle represents three mangoes, the solution must be 3 + 3 + 3 + 3 + 3 + 3, writing each 3 above one of the circles they had shaded.

Other students may draw a picture but divide a pie into six wedges. In the image below, one student shaded one wedge, noted five remaining wedges, and shaded one of them. She continued until she had shaded five of the six wedges. Finally, thinking about the sixth wedge, she said, "That's three."

Work backward:This strategy requires three steps: start at the end of the problem (the 3 remaining mangoes); reverse each of the steps in the problem, being careful to determine the amount at this step; and work the problem from end to beginning by performing the inverse operation at each step.

Applying these steps to the mangoes problem results in the following:

At the end 3 mangoes are left, representing one-half of the pile that the third Prince took. Thus, the third Prince had 6 mangoes before removing his half.

To determine how many the second Prince had before removing his third, we must realize that the 6 mangoes left after removal represent two‑thirds of the pile from which he took his third. Thus, 6 is two-thirds of the number in the pile, or 6 × 3/2 = 9, the number in the pile before removal.

By continuing backward in this manner, 9 mangoes represent three-fourths of the pile before the first Prince took his, so 9 × 4/3 = 12 mangoes were in the pile the first Prince used.

Similarly, the Queen's pile was 12 × 5/4, or 15, and the King's must have been 15 ×6/5, or 18, the answer to the problem.

The following illustrates one student's method for solving this problem:

"Six represents two-thirds of something, so one-third must be three. So to get three-thirds, you must add the six (for two-thirds) to three (for one-third) and you have nine mangoes." Then, going the next-backward step, he said, "Nine needs one-fourth" (his words, meaning that since nine is three-fourths of the previous amount, it "needs" another fourth of this amount added to it), "so nine is three-fourths:

divide by three (i.e., 9/3) and add this to nine, obtaining twelve." He continued quickly in this way to the final, correct solution.

Write an equation (use a variable): Some middle school students might try this approach, especially if they are flexible in their algebraic thinking. Let x be the number of mangoes in the bowl before any are removed.

Since the King removed (1/6)x, then x - (1/6)x mangoes are left after his removal. Thus, (5/6)x mangoes are left.

The Queen removed one-fifth of (5/6)x, so (5/6)x - (1/5)(5/6)x, or (4/6)x, mangoes are left after her removal.

The first Prince removed one-fourth of (4/6)x mangoes, so (4/6x - (1/4)(4/6)x, or (3/6)x, mangoes are left after the first Prince's removal.

The second Prince removed one-third of (3/6)x, so (3/6)x - (1/3) (3/6)x, or (2/6)x, mangoes are left.

Finally, the third Prince removed one-half of (2/6)x, leaving 3 mangoes, so (2/6)x - (1/2)(2/6)x = 1/6x = 3. Solving 1/6x = 3 results in x = 18.

Sailors and Coconuts

As time permits, pose the following problem to the students:

Three sailors were marooned on a deserted island that was also inhabited by a band of monkeys. The sailors worked all day to collect coconuts but were too tired that night to count them, They agreed to divide them equally the next morning.

During the night, one sailor woke up and decided to take his share. He found that he could make three equal piles, with one coconut left over, which he threw to the monkeys. Thereupon, he put his own share in a pile down the beach, and left the remainder in a single pile near where they all slept.

Later that night, the second sailor awoke and, likewise, decided to take his share of coconuts. He also was able to make three equal piles, with one coconut left over, which he threw to the monkeys.

Somewhat later, the third sailor awoke and did exactly the same thing with the remaining coconuts.

In the morning, all three sailors noticed that the pile was considerably smaller, but each thought that he knew why and said nothing. When they then divided what was left of the original pile of coconuts equally, each sailor received seven and one was left over, which they threw to the monkeys.

How many coconuts were in the original pile?

Distribute the The Sailors and Coconuts Problem activity sheet so students may see the text of the entire problem and have a place to show their work.

The Sailors and Coconuts Problem Activity Sheet

As in the previous problem, students should use various strategies to solve this problem. (The solution is 79 coconuts.)

Extensions

Suppose ten people take the "remaining" mangoes, just like in the original problem, that is, the first person takes one-tenth of the mangoes in the bowl, the second takes one-ninth of the remaining mangoes, the third takes one-eighth of the remaining ones, and so on, until only three are left. How many were in the original bowl?

Work a number of mango like problems starting with ten people, then nine people, then maybe only seven people. Figure out how many mangoes were originally involved in each problem. Then make a generalization that would enable you to tell how many mangoes were in the original bowl if three were left and you knew how many people removed mangoes.

NCTM Standards and Expectations

Algebra 6-8

Develop an initial conceptual understanding of different uses of variables.

Number & Operations 6-8

Develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use.

Use factors, multiples, prime factorization, and relatively prime numbers to solve problems.